HURWITZ THEOREM AND PARALLELIZABLE SPHERES FROM TENSOR ANALYSIS

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Abstract

By using tensor analysis, we find a connection between normed algebras and the parallelizability of the spheres S^1, S^3 and S^7. In this process, we discovered the analogue of Hurwitz theorem for curved spaces and a geometrical unified formalism for the metric and the torsion. In order to achieve these goals we first develope a proof of Hurwitz theorem based in tensor analysis. It turns out that in contrast to the doubling procedure and Clifford algebra mechanism, our proof is entirely based in tensor algebra applied to the normed algebra condition. From the tersor analysis point of view our proof is straightforward and short. We also discuss a possible connection between our formalism and the Cayley-Dickson algebras and Hopf maps.

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I. INTRODUCTION

It is known that normed algebras are closely related to supersymmetry[1−3] and super p-branes[4], and that these two theories require tensor analysis for their formulation. Therefore, it may be interesting to study normed algebras from the tensor analysis point of view. Moreover, normed algebras, among other things, are physically interesting because they are division algebras and in this context there are a number of interesting connections with fundamental physics. Let us just give some few examples about this fact. It has been shown[5] that a generalized instantons in eight dimensions fit inside the family of gauge-theoretical solitons associated to normed algebras. There is a deep relation between division algebras and superparticles (see ref. 6, 7 and references there in) and twistor formulation of a massless particles[8,9]. Finite Lorentz transformations of vectors in 10-dimensional Minkowski space have been studied[10] by means of division algebras. Finally, division algebras seem to be deeply related to the geometric structures of M-theory[11].

In this work, we show that tensor analysis can be used to give a straightforward connection between normed algebras and the paralellizability of the spheres S^1, S^3 and S^7. In the process of studing this connection, we discovered the analogue of Hurwitz theorem for curved spaces and a unified formalism for the metric and the torsion. Our strategy to achieve these goals was first to develope a proof of Hurwitz's theorem[12] based in tensor analysis. It turns out that this proof is essentially based on the composition law rewritten in tensor notation. From the point of view of tensor analysis, such a proof is short and straightforward. In fact, we do not even require to use the doubling procedure[12] or the Clifford algebra mechanism[13].

The plan of the article is as follows. In section II, we introduce tensor notation and a proof of Hurwitz theorem based in tensor analysis. In section III, we briefly review the Cartan-Shouten equations as presented by Gursey and Tze. In section IV, using the Gursey-Tze's procedure, we show a connection between our proof of Hurwitz theorem and the paralellizability of the spheres S^1, S^3 and S^7. We also prove that such a connection leads to

a generalization of Hurwitz theorem for curved spaces. In section V, we develope a unified formalism for the metric and the torsion. Finally, in section VI, we make a number of final comments and briefly outline a possible extension of the present work to the case of Cayley-Dickson algebras and Hopf maps.

II. AN ALTERNATIVE PROOF OF HURWITZ THEOREM

Let us start recalling the Hurwitz theorem:

Theorem (Hurwitz, 1898): *Every normed algebra with an identity is isomorphic to one of following four algebras: the real numbers, the complex numbers, the quaternions, and the Cayley (octonion) numbers.*

Proof (alternative): Consider a $N = d + 1$dimensional algebra $\mathcal{A}$ over the real numbers R. Let

$$e_0, e_1, ..., e_d \tag{1}$$

be a basis of $\mathcal{A}$, and let

$$A = A^0 e_0 + A^1 e_1 + ... + A^d e_d \tag{2}$$

be the representation of a vector $A \ \epsilon \ \mathcal{A}$ relative to this basis. Here, $A^0, A^1, ..., A^d \epsilon R$. Take the multiplication table in the form

$$e_i e_j = C^0_{ij} e_0 + C^1_{ij} e_1 + ... + C^d_{ij} e_d, \\ (i, j = 0, 1, ..., d), \tag{3}$$

where C^k_{ij} , the so-called structure constants, are real numbers (See, for instance, I. L. Kantor and A.S. Solodovnikov[12], S. Okubo[13], Abdel-Khalek[14], J. Adem[15], F. R. Cohen[16], Y. A. Drozd and V. V. Kirichenko[17].)

Assume that the basis (1) is orthonormal with bi-linear symmetric non-degenerate scalar product given by

$$< e_i \mid e_j >= \delta_{ij}, \tag{4}$$

where δ_{ij} is the so-called Kronecker delta, with $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

Assume the Einstein summation convention: if the same index appears twice, once as superscript and once as a subscript, then the index is summed over all possible values. This convention allows to write (2) and (3) as

$$A = A^i e_i, \tag{5}$$

and

$$e_i e_j = C^k_{ij} e_k, \tag{6}$$

respectively.

We shall assume that e_i transforms as covariant first-rank tensor

$$e'_i = \Lambda^j_i e_j, \tag{7}$$

where, in order to leave invariant (4), Λ^j_i satisfies the conditions $\det \Lambda^j_i \neq 0$ and $\Lambda^i_k \Lambda^j_l \delta_{ij} = \delta_{kl}$ and therefore Λ^j_i is an element of an orthogonal transformation $O(N)$. Since A is an invariant quantity, from (5) and (7) we find that A^i should transform as

$$A'^i = \Lambda^i_j A^j, \tag{8}$$

i.e. A^i is a contravariant first-rank tensor. While from (6) and (7) we find that C^k_{ij} transforms as

$$C'^r_{st} = \Lambda^r_k \Lambda^i_s \Lambda^j_t C^k_{ij}, \tag{9}$$

i.e. C^k_{ij} is a mixed third-rank tensor (twice covariant and once contravariant).

According to the multiplication table (6) the product $AB = D$ for A, B and $D \epsilon\ \mathcal{A}$ is given by

$$A^i B^j C^k_{ij} = D^k. \tag{10}$$

A normed algebra is an algebra in which the composition law

$$< AB \mid AB >=< A \mid A >< B \mid B > \tag{11}$$

holds for any $A,\ B\ \epsilon\ \mathcal{A}$. It can be shown that this expression is equivalent to (see, for instance, section 3.1 of ref. 13)

$$< AB \mid CD > + < CB \mid AD >= 2 < A \mid C >< B \mid D >, \tag{12}$$

where $A,\ B, C, D\ \epsilon\ \mathcal{A}$. Choosing

$$A \to e_i, B \to e_j, C \to e_m \text{ and } D \to e_n \tag{13}$$

we find that (12) leads to

$$< e_i e_j \mid e_m e_n > + < e_m e_j \mid e_i e_n >= 2 < e_i \mid e_m >< e_j \mid e_n > . \tag{14}$$

Using (4) and (6), from (14) we obtain the key formula

$$C^k_{ij} C^l_{mn} \delta_{kl} + C^k_{mj} C^l_{in} \delta_{kl} = 2\delta_{im}\delta_{jn}. \tag{15}$$

Note that, although at first sight it looks like, (15) is not a Clifford algebra. The reason for this is that, at this level, there are not any symmetries between the indices i, j and k of C^k_{ij}. In this work, the formula (15) shall play a central role. Note that when $D = 1$, this equation admits the solution $C^0_{00} = 1$. Therefore, in what follows we shall be mainly interested in solutions of (15) when $D \neq 1$.

Let e_0 be the identity of the algebra $\mathcal{A}$. Then, the multiplication table (6) implies

$$e_0 e_j = C^k_{0j} e_k = e_j \tag{16}$$

and

$$e_j e_0 = C^k_{j0} e_k = e_j. \tag{17}$$

From (16) we find

$$C^k_{0j} = \delta^k_j, \tag{18}$$

while from (17) we obtain

$$C^k_{j0} = \delta^k_j, \tag{19}$$

where δ^k_j is also a Kronecker delta.

Let us now split the formula (15) as follows:

$$C^k_{0j} C^l_{0n} \delta_{kl} + C^k_{0j} C^l_{0n} \delta_{kl} = 2\delta_{jn}, \tag{20}$$

$$C^k_{i0} C^l_{m0} \delta_{kl} + C^k_{m0} C^l_{i0} \delta_{kl} = 2\delta_{im}, \tag{21}$$

$$C^k_{0j} C^l_{an} \delta_{kl} + C^k_{aj} C^l_{0n} \delta_{kl} = 0, \tag{22}$$

$$C^k_{i0} C^l_{ma} \delta_{kl} + C^k_{m0} C^l_{ia} \delta_{kl} = 0, \tag{23}$$

$$C^0_{ab} C^0_{cd} + C^0_{cb} C^0_{ad} + C^e_{ab} C^f_{cd} \delta_{ef} + C^e_{cb} C^f_{ad} \delta_{ef} = 2\delta_{ac}\delta_{bd}, \tag{24}$$

where the indices a, b,..., etc run from 1 to d. Using (18) and (19) we note that the equations (20) and (21) are identities. Moreover, the expression (22) gives

$$C_{anj} + C_{ajn} = 0, \tag{25}$$

while (23) leads to

$$C_{mai} + C_{iam} = 0, \tag{26}$$

where $C_{mai} = C^l_{ma}\delta_{il}$, i.e. we raised and lowed indices with δ^{il} and δ_{il} respectively. From (25) we obtain

$$C_{ab0} + C_{a0b} = 0, \tag{27}$$

and

$$C_{abc} + C_{acb} = 0. \tag{28}$$

While from (26) we get

$$C_{ba0} + C_{0ab} = 0 \tag{29}$$

and

$$C_{bac} + C_{cab} = 0. \tag{30}$$

Thus, using (18) and (19), we have that either (27) or (30) implies that

$$C^0_{ab} = -\delta_{ab}, \tag{31}$$

while (28) and (30) mean that the quantity C_{abc} is completely antisymmetric.

Now, by substituting (31) into (24) we obtain

$$C^e_{ab} C^f_{cd} \delta_{ef} + C^e_{cb} C^f_{ad} \delta_{ef} = 2\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} - \delta_{cb}\delta_{ad}. \tag{32}$$

Multiplying this equation by δ^{ac} and $C^{ad}_g = \delta^{ae}\delta^{af} C_{gef}$ we find

$$\delta^{ac} C^e_{ab} C^f_{cd} \delta_{ef} = (d-1)\delta_{bd} \tag{33}$$

and

$$C^{ad}_g C^e_{ab} C^f_{cd} \delta_{ef} + C^{ad}_g C^e_{cb} C^f_{ad} \delta_{ef} = 3C_{gcb}, \tag{34}$$

respectively, where we used the fact that C_{abc} is completely antisymmetric. Moreover, using again the property that C_{abc} is completely antisymmetric, we find that (33) becomes

$$C^{ce}_b C_{dce} = (d-1)\delta_{bd}, \tag{35}$$

while substituting (33) into (34) we have

$$C^a_{dg} C^e_{ab} C^d_{ec} = (d-4) C_{gbc}. \tag{36}$$

Multiplying (35) by δ^{bd} we find the formula

$$C^{abc}C_{abc} = d(d-1), \tag{37}$$

which for $d=0$ and $d=1$, admits the solution $C_{abc}=0$. Moreover, for $d=3$ the formula (37) admits the solution $C_{abc}=\varepsilon_{abc}$, where ε_{abc} is the completely antisymmetric Levi-Civita symbol, with $\varepsilon_{123}=1$.

Let us define

$$G_{abc} \equiv C^{g}_{da}C^{e}_{gb}C^{d}_{ec}. \tag{38}$$

Since C_{abc} is completely antisymmetric, we find that G_{abc} is also completely antisymmetric. From (36) and (38) we find that

$$G^{abc}G_{abc} = (d-4)^2 C^{abc}C_{abc}, \tag{39}$$

which by virtue of (37) leads to

$$G^{abc}G_{abc} = d(d-1)(d-4)^2. \tag{40}$$

Substituting (38) into (40) we get

$$C^{ag}_{h}C^{br}_{g}C^{ch}_{r}C^{e}_{da}C^{f}_{eb}C^{d}_{fc} = d(d-1)(d-4)^2, \tag{41}$$

which can be rewritten in the form

$$C^{ag}_{h}C^{br}_{g}C^{e}_{da}C^{f}_{eb}(C^{ch}_{r}C^{d}_{fc}) = d(d-1)(d-4)^2. \tag{42}$$

So, considering (32) we find that (42) becomes

$$C^{ag}_{h}C^{br}_{g}C^{e}_{da}C^{f}_{eb}(2\delta_{rf}\delta^{hd} - \delta^{h}_{r}\delta^{d}_{f} - \delta^{d}_{r}\delta^{h}_{f} - C^{ch}_{f}C^{d}_{rc}) = d(d-1)(d-4)^2. \tag{43}$$

Now, using (35), (36) and (37) and the fact that C_{abc} is completely antisymmetric, we obtain

$$\begin{aligned} &C^{ag}_{h}C^{br}_{g}C^{e}_{da}C^{f}_{eb}(2\delta_{rf}\delta^{hd} - \delta^{h}_{r}\delta^{d}_{f} - \delta^{d}_{r}\delta^{h}_{f}) = \\ &= 2d(d-1)(d-1) - d(d-1)(d-1) + d(d-1)(d-4) \\ &= d(d-1)(2d-5), \end{aligned} \tag{44}$$

while, since with respect to the indices a and h the quantity C^{ag}_h is antisymmetric and the tensor $(C^e_{da}C^f_{eb}C^{ch}_f C^d_{rc})$ is symmetric, we get

$$C^{ag}_h C^{br}_g C^e_{da}C^f_{eb}(C^{ch}_f C^d_{rc}) = C^{ag}_h C^{br}_g (C^e_{da}C^f_{eb}C^{ch}_f C^d_{rc}) \equiv 0. \tag{45}$$

Thus, by substituting the results (44) and (45) into (43), we discover the equation

$$d(d-1)(2d-5) = d(d-1)(d-4)^2, \tag{46}$$

which can be rewritten in the form

$$d(d-1)(d-3)(d-7) = 0. \tag{47}$$

The only solutions for this equation are $d = 0, 1, 3$ and 7. Therefore, we have shown that the equation (15) has solution only for $D = 1, 2, 4$ and 8. This implies that normed algebras with unit element are only possible in these dimensions.

We have yet to show that the cases $D = 1, D = 2, D = 4$ and $D = 8$ correspond to real, complex, quaternion and octonion algebras, respectively. The case $D = 1$ is trivial since for any $A \epsilon \mathcal{A}$, we have $A = A^0 e_0$, where $A^0 \epsilon\ R$. For the case $D = 2$, we have $C_{abc} = 0$, $C^0_{ab} = -\delta_{ab}$, $C^s_{n0} = \delta^s_n$ and $C^s_{0n} = \delta^s_n$. These values of the structure constants determine the algebra of complex numbers. While, for the case $D = 4$, we have the solution of (32) $C_{abc} = \varepsilon_{abc}$, $C^0_{ab} = -\delta_{ab}$, $C^s_{n0} = \delta^s_n$ and $C^s_{0n} = \delta^s_n$. It is well known that these values of the structure constants determine the algebra of quaternions. Finally, for the case $D = 8$ we have $C^0_{ab} = -\delta_{ab}$, $C^s_{n0} = \delta^s_n$ and $C^s_{0n} = \delta^s_n$. Now, take the structure constants as $C_{abc} = \Xi_{abc}$, where Ξ_{abc} is a completely antisymmetric Levi-Civita symbol, with $\Xi_{abc} = 1$, for the following values of the indices (a, b, c):

$$(1,2,3), (5,1,6), (6,2,4), (4,3,5), (1,7,4), (3,7,6) \text{ and } (2,7,5). \tag{48}$$

In fact, these values of the structure constants determine the algebra of octonions. One can verify by straightforward, but tedious, computation that, in fact for $d = 7$, these values for the structure constants give a solution of (32).

It is known that by definition two $(d+1)-$ dimensional algebras $\mathcal{A}'$ and $\mathcal{A}$ are said to be isomorphic if they have bases with identical multiplication table. Therefore, it remains to show that any other solution is isomorphic to one of the above four solutions corresponding to the real numbers, the complex numbers, the quaternions and the octonions. For this purpose it is convenient to set $e'_0 = e_0$. So that from the transformation rule (7) we find that $\Lambda^0_0 = 1$ and $\Lambda^a_0 = 0$. Thus, from the relation $\Lambda^i_k \Lambda^j_l \delta_{ij} = \delta_{kl}$, which leave invariant the scalar product (4), we find that $\Lambda^0_a = 0$ and therefore we have now the relation $\Lambda^c_a \Lambda^d_b \delta_{cd} = \delta_{ab}$ which leaves invariant the scalar product $< e_a \mid e_b >= \delta_{ab}$. Consequently, we have that Λ^d_a is an element of $O(d) = O(D-1)$ which is a subgroup of $O(D)$. Note that the property $\det \Lambda^i_j \neq 0$ now becomes $\det \Lambda^d_a \neq 0$. Clearly, the transformation Λ^d_a acts over elements of the sub-vector space $\mathcal{A}_0$ of $\mathcal{A}$ defined by $\mathcal{A}_0 = \{A \mid < A \mid e_0 >= 0, A \epsilon \mathcal{A}\}$, with Dim $\mathcal{A}_0 = d$. In fact, we can write $\mathcal{A} = \lambda e_0 \oplus \mathcal{A}_0$, with $\lambda \epsilon R$.

Thus, we find that the structure constants C_{abc} transform according to

$$C'_{abc} = \Lambda^d_a \Lambda^e_b \Lambda^f_c C_{def}. \tag{49}$$

Note that, since $\Lambda^d_a \Lambda^e_b \delta_{de} = \delta_{ab}$, if C_{abc} is a solution of (32), then C'_{abc} is also a solution.

The transformation (49) has the important property that $C_{def} = 0$ if and only if $C'_{abc} = 0$. Therefore for real numbers, as well as for complex numbers, the two algebras $\mathcal{A}'$ and $\mathcal{A}$ are isomorphic. For quaternions take $C_{def} = \varepsilon_{def}$ then (49) implies that $C'_{abc} = \Lambda \varepsilon_{abc}$, $\Lambda \equiv \det \Lambda^d_a$. Thus, if $C_{def} = \varepsilon_{def}$ is a solution of (32) we have that $C'_{abc} = \Lambda \varepsilon_{abc}$ is also a solution. Therefore, for $D = 4$ any solution of (32) is isomorphic to the quaternionic solution, corresponding to $C_{abc} = \varepsilon_{abc}$. Similarly, for octonions applying (49) to the completely antisymmetric symbol Ξ_{abc} we find that $\Xi'_{abc} = \Lambda \Xi_{abc}$, where the values of the indices (a, b, c) are given in (48).

Therefore, we have shown that up to isomorphism the dimensions $D = 1, 2, 4$ and 8 correspond to real, complex, quaternion and octonion algebras, respectively. And in this way using the mathematical tool of tensor analysis we have given an alternative proof of Hurwitz theorem. It is an interesting and remarkable fact that without using doubling

procedure (see ref. 12) or Clifford algebra mechanism (see ref. 13) our proof has been based almost completely in tensor algebra applied to the formula (15).

III. CARTAN-SHOUTEN EQUATIONS

Define the metric tensor by

$$g_{ab} = \delta_{cd} h_a^{(c)} h_b^{(d)}, \tag{50}$$

where $h_a^{(c)} = h_a^{(c)}(x^{b)})$ is a vielbein field. Here, x^a is a coordinate patch of the geometrical sphere S^d. The quantities C_{abc} can now be related to the S^d torsion in the form

$$T_{abc} = r^{-1} C_{efg} h_a^{(e)} h_b^{(f)} h_c^{(g)}, \tag{51}$$

where r is the radius of S^d. Using (35), (36), (50) and (51) we find that the torsion T_{abc} satisfies the equations:

$$T_a^{cd} T_{bcd} = (d-1) r^{-2} g_{ab}, \tag{52}$$

and

$$T_{ea}^{d} T_{db}^{f} T_{fc}^{e} = (d-4) r^{-2} T_{abc.} \tag{53}$$

We recognize these expressions as the Cartan-Schouten equations[18]which as Gursey and Tze[19] noted, are mere septad-dressed, i.e. covariant forms of the algebraic identities (35) and (36). It is well known that these equations are closely related to the parallelizability of S^1, S^3 and S^7 (see ref. 13). In fact, the equations (52) and (53) can be derived by adding to the riemannian symmetric connection Γ_{ab}^c the totally antisymmetric torsion tensor T_{ab}^c and ”flattening” the space in the sense that

$$\mathcal{R}_{bcd}^{a}(\{\Omega_{ab}^c\}) = 0, \tag{54}$$

where

$$\mathcal{R}_{bcd}^{\;a} = \partial_c \Omega_{bd}^a - \partial_d \Omega_{bc}^a + \Omega_{ec}^a \Omega_{bd}^e - \Omega_{ed}^a \Omega_{bc}^e, \tag{55}$$

with

$$\Omega_{ab}^c = \Gamma_{ab}^c + T_{ab}^c. \tag{56}$$

For our purpose it is convenient to show explicitly that in fact the equations (52) and (53) follow from (54)-(56). By substituting (56) into (54) we find

$$0 = R_{bcd}^{\;a} + D_c T_{bd}^a - D_d T_{bc}^a + T_{ec}^a T_{bd}^e - T_{ed}^a T_{bc}^e, \tag{57}$$

Here, D_c denotes a covariant derivative with Γ_{ab}^c as a connection and

$$R_{bcd}^{\;a} = \partial_c \Gamma_{bd}^a - \partial_d \Gamma_{bc}^a + \Gamma_{ec}^a \Gamma_{bd}^e - \Gamma_{ed}^a \Gamma_{bc}^e. \tag{58}$$

Using in (57) the cyclic identities for $R^{\;a}_{\;\;bcd}$ leads to

$$D_c T_{bda} = T_{e[bd} T_{a]c}^e, \tag{59}$$

where

$$T_{e[bd} T_{a]c}^e \equiv \frac{1}{3}\{T_{ebd} T_{ac}^e + T_{eab} T_{dc}^e + T_{eda} T_{bc}^e\}. \tag{60}$$

Substituting (59) into (57) we obtain

$$R_{abcd} = T_{eab} T_{cd}^e - T_{e[ab} T_{c]d}^e. \tag{61}$$

For the sphere S^d we have

$$R_{abcd} = \frac{1}{r^2}(g_{ac} g_{bd} - g_{ad} g_{bc}). \tag{62}$$

and therefore we get the equation

$$\frac{1}{r^2}(g_{ac} g_{bd} - g_{ad} g_{bc}) = T_{eab} T_{cd}^e - T_{e[ab} T_{c]d}^e. \tag{63}$$

Contracting in (63) with g^{ac} leads to first Cartan-Shouten equation (52), while contracting (63) with T_f^{ac} leads to the second Cartan-Shouten equation (53).

IV. NORMED ALGEBRAS AND PARALLELIZABILITY OF S^1, S^3 and S^7

Let us start ‘undressing’ (63). Using (50) and (51) we find

$$(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) = C_{eab}C^{e}_{cd} - C_{e[ab}C^{e}_{c]d}. \tag{64}$$

We shall show that this formula is equivalent to the formula (32). For this purpose, let us rewrite formula (32) in form.

$$2\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{cb} = C^{e}_{ab}C^{f}_{cd}\delta_{ef} + C^{e}_{cb}C^{f}_{ad}\delta_{ef}. \tag{65}$$

Let us first show that (64) implies (65). Making the change of indices $a \to c$ and $c \to a$ in (64) we find

$$(\delta_{ca}\delta_{bd} - \delta_{cd}\delta_{ba}) = C_{ecb}C^{e}_{ad} - C_{e[cb}C^{e}_{a]d}. \tag{66}$$

By adding (64) and (66) one easily obtains (65).

Let us now show that (65) implies (64). Let us start writing (65) in the form

$$C^{e}_{ab}C^{f}_{cd}\delta_{ef} - (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{cb}) + C^{e}_{cb}C^{f}_{ad}\delta_{ef} - (\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd}) = 0. \tag{67}$$

This expression suggests to define

$$F_{abcd} \equiv C^{e}_{ab}C^{f}_{cd}\delta_{ef} - (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{cb}). \tag{68}$$

Therefore (67) gives

$$F_{abcd} + F_{cbad} = 0. \tag{69}$$

Thus, considering that C^{e}_{ab} is completely antisymmetric, from (68) and (69) we discover that F_{abcd} is also completely antisymmetric. Using this important cyclic property for F_{abcd} it is not difficult to show that

$$F_{abcd} = C_{e[ab}C^{e}_{c]d}. \tag{70}$$

Substituting this result into (68) lead us back to (64). Thus, we have proved the equivalence between (64) and (65).

With this equivalence at hand we have a number of interesting observations. First, since in section II we showed that (65) (or (32)) admits solution only for dimensions $d = 1, 3$ and 7 we have that (64) admits solution only in these dimensions. But, since (64) is the necessary and sufficient condition for the existence of parallelism in S^d, this means that we have found an alternative proof of the fact that only the spheres S^1,S^3 and S^7 are parallelizables. Second, in section II we proved that (65) is a consequence of the normed condition (15) (or equivalent of (11)), while in section III we proved that (64) is a consequence of the paralizability condition (54). Therefore, we have find a new bridge between normed algebras and parallelizable spheres. This link can be more transparent if using (50) and (51) we dress (65) in the form

$$\frac{1}{r^2}(2g_{ac}g_{bd} - g_{ab}g_{cd} - g_{ad}g_{cb}) = T^e_{ab}T^f_{cd}g_{ef} + T^e_{cb}T^f_{ad}g_{ef}. \tag{71}$$

Of course, the equations (63) and (71) are equivalent. So, from (65) we can derive (71) which in turn leads to the formula (63). Going backwards from (63) we get (61). Therefore, we have shown that normed algebra condition (65) implies the parallelizable condition (61). Similarly, we can show that the parallelizable condition (61) implies the composition law (65). Moreover, (31) and (71) suggest to define

$$T^0_{ab} \equiv -r^{-1}g_{ab}. \tag{72}$$

Thus, using (72) we find that (71) can be written in the form

$$T^k_{ab}T^l_{cd}g_{kl} + T^k_{cb}T^l_{ad}g_{kl} = \frac{2}{r^2}g_{ac}g_{bd}. \tag{73}$$

where we recall that the indices m and n run from 0 to d. Setting $g_{00} = 1$ and $g_{0a} = 0$ we obtain (72) from (73). If we now take $T^k_{0j} = \delta^k_j$ and $T^k_{j0} = \delta^k_j$, then we can generalize (80) in the form

$$T^k_{ij}T^l_{mn}g_{kl} + T^k_{mj}T^l_{in}g_{kl} = \frac{2}{r^2}g_{im}g_{jn}. \tag{74}$$

If we now introduce a basis h_m such that

$$< h_m \mid h_n >= g_{mn}, \tag{75}$$

and

$$h_m h_n = T^k_{mn} h_k, \tag{76}$$

we find that (74) leads to a generalization of (14)

$$< h_i h_j \mid h_m h_n > + < h_m h_j \mid h_i h_n >= \frac{2}{r^2} < h_i \mid h_m >< h_j \mid h_n > . \tag{77}$$

Clearly, this expression implies the generalized composition law condition

$$< AB \mid AB >= \frac{1}{r^2} < A \mid A >< B \mid B >, \tag{78}$$

where $A = A^i h_i$.

The r^2 in the right hand side of (74) remind us that our construction is valid for spheres. However, the equation (74) allows an straightforward generalization. In fact, let us prove the theorem ($\star$) below:

Before going into the details of the theorem let us define a 'curved' space as a space in which (75) and (74) hold, with $g_{00} = 1$, $g_{0a} = 0$ and $g_{ab} = g_{ab}(x^i) = \eta_{cd} h^{(c)}_a(x^i) h^{(d)}_b(x^i)$, where the flat metric η_{cd} is diagonal and has an arbitrary signature and $T^k_{ij} = T^k_{ij}(x^i)$.

Theorem ($\star$): *The possible dimensions D of any real normed algebra over a 'curved' space with an identity are limited to only 1, 2, 4 and 8.*

Proof: Let us write the composition law as follows:

$$< h_i h_j \mid h_m h_n > + < h_m h_j \mid h_i h_n >= 2 < h_i \mid h_m >< h_j \mid h_n > . \tag{79}$$

By virtue of (75) and (76) we find that (79) can be written as

$$T^k_{ij} T^l_{mn} g_{kl} + T^k_{mj} T^l_{in} g_{kl} = 2 g_{im} g_{jn}. \tag{80}$$

Taking h_0 as the identity with the properties that

$$< h_0 \mid h_0 >= g_{00} = 1, \tag{81}$$

and

$$< h_0 \mid h_a >= g_{0a} = 0, \tag{82}$$

and following the same procedure as in section II, we find

$$T^k_{0j} = T^k_{j0} = \delta^k_j, \tag{83}$$

$$T^0_{ab} = -g_{ab}, \tag{84}$$

$$(2g_{ac}g_{bd} - g_{ab}g_{cd} - g_{ad}g_{cb}) = T^e_{ab}T^f_{cd}g_{ef} + T^e_{cb}T^f_{ad}g_{ef} \tag{85}$$

with the property that T^e_{ab} is completely antisymmetric. The rest of the story is similar to section II after formula (32). We find that (85) has solution only if $d = 1, 3$ and 7. Note that in this result g_{ab} may be the metric not only for the spheres S^1,S^3 and S^7, but also the metric of any curved space. Moreover, in 'flat' space g_{ab} may have an arbitrary signature. In particular for $D = 4$ we could associate to g_{ij} the signature $(g_{ij}) = diag(1, 1, 1, -1)$ which correspond to Minkowski signature. Note also that T^k_{ij} unifies the metric g_{ab} and the torsion T^e_{ab}.

Summarizing, we have proved not only an equivalence between the Hurwitz theorem for normed algebras and Cartan-Shouten theorem for parallelizable spheres, but also the theorem ($\star$).

V. UNIFIED FORMALISM OF THE METRIC AND THE TORSION

In the previous section, in the context of normed algebras, we showed that makes sense to unify the metric and the torsion in just one mathematical object: the third-rank tensor T^k_{ij} . A natural question is to see what is the geometry induced by T^k_{ij}. In this section we show that

from the vanishing of the Riemann tensor associated to such a third-rank tensor it follows the metricity condition and the Cartan-Shouten equations for homogeneous spacetimes.

Consider the equation

$$\mathcal{R}^{\,i}_{jkl}(\Omega^{i}_{jk}) = 0, \tag{86}$$

where

$$\mathcal{R}^{i}_{jkl} = \partial_k \Omega^{i}_{jl} - \partial_l \Omega^{i}_{jk} + \Omega^{i}_{mk}\Omega^{m}_{jl} - \Omega^{i}_{el}\Omega^{e}_{jk} \tag{87}$$

and

$$\Omega^{i}_{jk} = \Gamma^{i}_{jk} + T^{i}_{jk}. \tag{88}$$

These equations are, of course the analogue of the parallelizability conditions (54)-(56). Let us see what are the consequences of (86)-(88). For this purpose let us assume that T^{i}_{jk} satisfies (83) and (84) and let us set

$$\Gamma^{0}_{jk} = 0 \text{ and } \Gamma^{i}_{0k} = 0. \tag{89}$$

Thus, the non-vanishing terms of Ω^{i}_{jk} are

$$\Omega^{c}_{ab} = \Gamma^{c}_{ab} + T^{c}_{ab}, \tag{90}$$

$$\Omega^{0}_{ab} = -g_{ab}, \tag{91}$$

$$\Omega^{a}_{0b} = \delta^{a}_{b} \tag{92}$$

and

$$\Omega^{0}_{00} = 1. \tag{93}$$

At this stage it is important to note that (90)-(93) could also be obtained if instead of (83), (84) and (89) we set $T^{0}_{jk} = 0, T^{i}_{0k} = 0, \Gamma^{0}_{0k} = 0$, $\Gamma^{0}_{00} = 1$, $\Gamma^{a}_{0b} = \delta^{a}_{b}$and $\Gamma^{0}_{ab} = -g_{ab}$. However,

with this choice, the connection between (80) and (85) will be lost. This connection is, of course, important to make contact with the normed algebras for 'curved' space discussed in the previous section. It is worth mentioning that the formulae (83), (84) and (89) can be understood as an anzats in the sense of Kaluza-Klein theory.

Let us split (87) in the form

$$\mathcal{R}^0_{abc} = \partial_b \Omega^0_{ac} - \partial_c \Omega^0_{ab} + \Omega^0_{0b}\Omega^0_{ac} + \Omega^0_{db}\Omega^d_{ac} - \Omega^0_{0c}\Omega^0_{ab} - \Omega^0_{dc}\Omega^d_{ab}, \tag{94}$$

$$\mathcal{R}^0_{a0c} = \partial_0 \Omega^0_{ac} - \partial_c \Omega^0_{a0} + \Omega^0_{00}\Omega^0_{ac} + \Omega^0_{d0}\Omega^d_{ac} - \Omega^0_{0c}\Omega^0_{a0} - \Omega^0_{dc}\Omega^d_{a0}, \tag{95}$$

$$\mathcal{R}^b_{a0c} = \partial_0 \Omega^b_{ac} - \partial_c \Omega^b_{a0} + \Omega^b_{00}\Omega^0_{ac} + \Omega^b_{d0}\Omega^d_{ac} - \Omega^b_{0c}\Omega^0_{a0} - \Omega^b_{dc}\Omega^d_{a0} \tag{96}$$

and

$$\mathcal{R}^d_{abc} = \partial_b \Omega^d_{ac} - \partial_c \Omega^d_{ab} + \Omega^d_{0b}\Omega^0_{ac} + \Omega^d_{eb}\Omega^e_{ac} - \Omega^d_{0c}\Omega^0_{ab} - \Omega^d_{ec}\Omega^e_{ab}. \tag{97}$$

Considering (90)-(93) it is straightforward to see that these formulae are reduced to

$$\mathcal{R}^0_{abc} = -\partial_b g_{ac} + \partial_c g_{ab} - g_{db}\Gamma^d_{ac} - g_{db}T^d_{ac} + g_{dc}\Gamma^d_{ab} + g_{dc}T^d_{ab}, \tag{98}$$

$$\mathcal{R}^0_{a0c} = -\partial_0 g_{ac}, \tag{99}$$

$$\mathcal{R}^b_{a0c} = \partial_0 \Omega^b_{ac} \tag{100}$$

and

$$\mathcal{R}^d_{abc} = R^d_{abc} - \delta^d_b g_{ac} + \delta^d_c g_{ab} + D_b T^d_{ac} - D_c T^d_{ab} + T^d_{eb}T^e_{ac} - T^d_{ec}T^e_{ab}, \tag{101}$$

respectively. Here, we recall that D_a denotes a covariant derivative in terms of Γ^d_{ac}. The Equation (86) implies $\partial_0 g_{ac} = 0$ and $\partial_0 \Omega^b_{ac} = 0$, that is , g_{ac} and Ω^b_{ac} are independents of x^0. This result remind us the dimensional reduction procedure in Kaluza-Klein theory.

Let us now focus in (98). Since T^d_{ac} is completely antisymmetric, using (86) the equation (98) leads to

$$\partial_b g_{ac} - \partial_c g_{ab} + g_{db}\Gamma^d_{ac} - g_{dc}\Gamma^d_{ab} = 2T_{cab}, \tag{102}$$

Combining the indices in (102) we also get

$$\partial_a g_{bc} - \partial_c g_{ba} + g_{da}\Gamma^d_{bc} - g_{dc}\Gamma^d_{ba} = 2T_{cba}, \tag{103}$$

Thus, adding these two expressions we obtain the equation

$$\partial_b g_{ac} + \partial_a g_{bc} - 2\partial_c g_{ab} + g_{db}\Gamma^d_{ac} + g_{da}\Gamma^d_{bc} - 2g_{dc}\Gamma^d_{ab} = 0, \tag{104}$$

whose solution is

$$\Gamma_{cab} = \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}). \tag{105}$$

We recognize in this expression the traditional definition of Christoffel symbols. Moreover, it is well known that this expression is equivalent to the metricity condition

$$D_a g_{bc} = 0, \tag{106}$$

Therefore, we have shown that the metricity condition follows from the equation (98).

Consider now the expression (101). Using (86) we get

$$R^d_{abc} - \delta^d_b g_{ac} + \delta^d_c g_{ab} + D_b T^d_{ac} - D_c T^d_{ab} + T^d_{eb} T^e_{ac} - T^d_{ec} T^e_{ab} = 0. \tag{107}$$

For a homogenous space we have

$$R^d{}_{abc} = \gamma(\delta^d_b g_{ac} - \delta^d_c g_{ab}), \tag{108}$$

where γ is a constant. Thus, introducing a new constant $\gamma' = \gamma - 1$ the equation (107) becomes

$$\gamma'(\delta^d_b g_{ac} - \delta^d_c g_{ab}) + D_b T^d_{ac} - D_c T^d_{ab} + T^d_{eb} T^e_{ac} - T^d_{ec} T^e_{ab} = 0. \tag{109}$$

We recognize this expression as the equation (57). Hence, it is straightforward to prove that expression (109) implies the Cartan-Shouten equations. Therefore, we have shown that the metricity condition (106) and the Cartan-Shouten equations follow from (86)-(88).

VI. COMMENTS

It is known that Hurwitz theorem is closely related to the generalized Frobenius theorem (see ref. 12 and references there in): *Every alternative division algebra is isomorphic to one of the following : the algebra of real numbers, the algebra of complex numbers, the quaternions, and the Cayley numbers.* In fact, using Hurwitz theorem the generalized Frobenius theorem can be proved . Therefore, our procedure also gives an alternative proof of such a generalized theorem. Let just show how our procedure can be used to clarify such a relation.

Alternative algebras can be defined by means of the associator

$$(e_i, e_j, e_k) = (e_i e_j)e_k - e_i(e_j e_k) \equiv \mathcal{F}^l_{ijk} e_l. \tag{110}$$

In fact, if $\mathcal{F}_{ijkl} = \delta_{lm}\mathcal{F}^m_{ijk}$ is completely antisymmetric for exchanges of any two indices then the algebra is called alternative. Using (4) and (6) one can show that (86) is equivalent to

$$\mathcal{F}_{ijkl} = C^m_{ij} C_{mkl} - C^m_{jk} C_{iml}. \tag{111}$$

Now, in section II we showed that normed algebra with an identity implies that $C^m_{0j} = \delta^m_j, C^m_{j0} = \delta^m_j$ and $C^0_{ab} = -\delta_{ab}$ and that C^c_{ab} is a completely antisymmetric quantity satisfying (32). From these conditions it follows that

$$\mathcal{F}_{abcd} = C^m_{ab} C_{mcd} - C^m_{bc} C_{amd}. \tag{112}$$

are the only non-vanishing components of $\mathcal{F}_{ijkl}$. Using (32), it is not difficult to show that this expression leads to

$$\mathcal{F}_{abcd} = 2\{C^e_{ab} C^f_{cd} \delta_{ef} - (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{cb})\} = 2F_{abcd}, \tag{113}$$

where F_{abcd} has been defined in (68). In section IV, we proved that (113) can be obtained from any normed algebra. Therefore since (111) is equivalent to (113) we have shown that a normed algebra with an identity is alternative algebra. The fact that a normed algebra is

a division algebra can be proved directly from the composition law $< AB \mid AB >=< A \mid A >< B \mid B >$. Indeed, if $AB = 0$ the composition law implies that $< A \mid A >= 0$ or $< B \mid B >= 0$, which means that $A = 0$ or $B = 0$. Thus, our procedure based in tensor analysis gives a straightforward proof of the fact that a normed algebra with an identity is an alternative division algebra.

It may be interesting to apply the procedure presented in this paper in different contexts. For instance, it may be helpful to through some light on the Blencowe-Duff conjecture[4]: Do the four forces in Nature correspond to the four division algebras? In fact, part of the motivation of this work arose as an effort for answering this question. It is known[20] that using an algebraic topology called K-theory[21] we find that the only dimensions for division algebras structures on Euclidean spaces are again 1, 2, 4, and 8. Therefore, it may be also interesting to relate the present work to K-theory. Moreover, it is known that Englert's solution of eleven dimensional supergravity achieves the riemannian curvatureless but torsion-full Cartan geometries of absolute parallelism on S^7. Therefore, it may be interesting to see if the present work may shed some light to clarify some aspects of eleven dimensional supergravity which, as it is known, is the low energy limit theory of M-theory[22−27]. It also seems interesting to see if tensor analysis may be useful to study the zero divisors of Cayley-Dickson algebras[28]and Hopf maps. Let briefly outline this last possibility.

The Cayley-Dickson algebras are defined by the product

$$AB = (A_1 B_1 - \bar{A}_2 B_2, B_2 A_1 + A_2 \bar{B}_1), \tag{114}$$

where $A = (A_1, A_2)$ and $B = (B_1, B_2)$ are in $R^{2^n} = R^{2^{n-1}} \times R^{2^{n-1}}$ and $\bar{A} = (\bar{A}_1, -A_2)$. Let us denote an algebra with this structure by A_n. It is found that A_0 =real numbers R, $A_1 =$ complex numbers, A_2 =quaternions and $A_3 =$ octonions. A Hopf map is defined as

$$F_n : A_n \times A_n \to A_n \times A_o \tag{115}$$

$$F_n = (2AB, < B \mid B > - < A \mid A >).$$

Consider the multiplication table

$$e_i e_j = D_{ij}^{\alpha} e_{\alpha}, \tag{116}$$

where D_{ij}^{α} are the structure constants, with $i, j, k = 0, 1, ..., 2^n - 1$ and $\alpha, \beta = 0, 1, ..., 2^n$. Suppose D_{ij}^{α} satisfies the conditions

$$D_{ij}^{2^n} = \delta_{ij} \tag{117}$$

and

$$D_{ij}^{k} = -D_{ji}^{k}, \tag{118}$$

where in (117) we set $\alpha = 2^n$. Now, take $H^i = B^i + A^i$ and $G^i = B^i - A^i$ and consider the product

$$F^{\alpha} = H^i G^j D_{ij}^{\alpha}. \tag{119}$$

Using (117) and (118) we find

$$F^{2^n} =< B \mid B > - < A \mid A > \tag{120}$$

and

$$F^k = 2A^i B^j D_{ij}^k. \tag{121}$$

Therefore, F^{α} defined in (119) reproduces the Hopf map. It remains to find the relation between D_{ij}^k and the Cayley-Dickson product. At this respect, our final goal is to see if our procedure may shed some light on the Hopf maps

$$\begin{gathered} F_0 : S^1 \to S^1, \\ \\ F_1 : S^3 \to S^2, \\ \\ F_2 : S^7 \to S^4, \\ \\ F_3 : S^{15} \to S^8. \end{gathered} \tag{122}$$

which have a certain topological invariant , the Hopf invariant, equal to one.

Finally, it may be interesting to find the connection between the present paper and the Wolf's works of references 29 and 30, in which the Cartan-Shouten formalism is generalized to the case of non-Euclidean spaces. Moreover, a possible connection between our procedure and flexible Malcev-admissible algebras (see references 31, 32 and 33 and references there in) may deserve further research.

REFERENCES

[1] T. Kugo and P. Townsend, Nucl. Phys. **B221,** 357(1983).

[2] J. M. Evans, Nucl. Phys. **B298,** 92 (1988).

[3] G. Sierra, Class. Quantum Grav. **4,** 227 (1987).

[4] M. P. Blencowe and M. J. Duff, Nucl. Phys. **B310,** 387 (1988).

[5] J. M. Figueroa-O'Farril, J. Geom. Phys. **32**, 227 (1999).

[6] T. Kimura, I. Oda, Prog. Theor. Phys. **80**, 1 (1988).

[7] I. Oda, T. Kimura and A. Nakamura, Prog. Theor. Phys. **80**, 367 (1988).

[8] I. Bengtsson, Nucl. Phys. **B302**, 81 (1988).

[9] R. Foot and G. C. Joshi, Lett. Math. Phys. **16**, 77 (1988).

[10] C. A. Monogue and J. Schray, J. Math. Phys. **34**, 3746 (1993).

[11] C. M. Hull, JHEP, **9811**, 017 (1998); hep-th/9807127.

[12] I.L. Kantor and A.S. Solodovnikov, *Hypercomplex Numbers; An Elementary Introduction to Algebras (*Springer-Verlag New York, 1989).

[13] S. Okubo, *Introduction to Octonion and Other Non-Associative Algebras* (Cambridge University Press, 1995).

[14] K. Abdel-Khalek, Int. J. Mod. Phys. **A13,** 569 (1998); hep-th/9704049.

[15] J. Adem, Obra Matemática, El Colegio Nacional, México, (1992).

[16] F. R. Cohen, Bol. Soc. Mat. Mex. **37** 55 (1992).

[17] Y. A. Drozd and V. V. Kirichenko, *Finite Dimensional Algebras* (Springer-Verlag, 1994).

[18] E. Cartan and J. A. Schouten, Proc. K. Akad. Wet. Amsterdam **29,** 803 (1926).

[19] F. Gursey and C. Tze, Phys. Lett. **B127,** 191 (1983).

[20] N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, 1970).

[21] M. F. Atiyah. *K-theory* (W. A. Benjamin Inc., 1979).

[22] P. K. Townsend, *Four Lectures on M Theory*, ICTP Summer School on High Energy Physics and Cosmology, Trieste, June 1996; hep-th/9612121.

[23] M. J. Duff, Int. J. Mod. Phys. **A11,** 5623 (1996); hep-th/9608117.

[24] P. Horava and E. Witten, Nucl. Phys. **B460,** 506 (1996).

[25] M. J. Duff, R. R. Khuri and J. X. Lu, Phys. Rep. **259,** 213 (1995).

[26] E. Witten, Nucl. Phys. **B463,** 383 (1996).

[27] J. H. Schwarz, Nucl. Phys. Proc. Suppl. **B55,** 1 (1997); hep-th/9607201.

[28] G. Moreno, *The zero divisors of Cayley-Dickson algebras,* Reporte Interno CINVESTAV, No. **195**, (1995); G_2 *como Solución a un Problema de Teoría de Singularidades,* Ap. Mat. Ser. Com. **20**, 129 (1997).

[29] J. A: Wolf, J. Diff. Geom. **6** (1972) 317.

[30] J. A: Wolf, J. Diff. Geom. **7** (1972) 19.

[31] H. C. Myung, *Malcev admissible algebras* (Birkhauser, Boston, 1986).

[32] M. L. El-Mallah, Archiv. der Math. **49** 16 (1987); and **51** 39 (1988) .

[33] S. Okubo, Hadronic J. **4,** 216 (1981).